

On a conjecture by Blocki-Zwonek
by

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Abstract. This paper gives a counterexample to a conjecture by Blocki and Zwonek on the area of sublevel sets of the Green's function.

1. INTRODUCTION

In [1] Blocki and Zwonek make the following conjecture:

Conjecture 1.1. If Ω is a pseudoconvex domain in \mathbb{C}^n , then the function

$$t \rightarrow \log \lambda(\{G_{\Omega,w} < t\})$$

is convex.

Here w is any given point in Ω and G denotes the pluricomplex Green function with pole at w . Also λ denotes Lebesgue measure.

In this note we provide a counterexample in \mathbb{C} . The example is constructed in the next section. It is a triply connected domain where the Green function has a higher order critical point.

2. THE EXAMPLE

Lemma 2.1. *There exists a bounded domain $\Omega \subset \mathbb{C}$ and a point $q \in \Omega$ such that the Green function G with pole at q has a critical point $p \in \Omega$ so that $G(z) - G(p) = \mathcal{O}(|z - p|^3)$ in a neighborhood of p . The domain is triply connected with boundary consisting of three simply closed real analytic smooth curves.*

Proof. We construct the domain Ω by modifying domains in steps:

$$\Omega_1, \Omega_2, \Omega_3, \Omega_4 = \Omega.$$

First, we define $\Omega_1 \subset \mathbb{P}^1$. The point ∞ will be an interior point. The constant $r = e^{-1/3}$, $0 < r < 1$.

$$\Omega_1 := \{\eta \in \mathbb{P}^1; |\eta + 1| > r.\}$$

So Ω_1 is a topological disc. $\eta = 0 \in \Omega_1$. The complement of Ω_1 is the closed disc centered at $\eta = -1$ with radius r .

Next we let

$$\Omega_2 := \{\tau \in \mathbb{P}^1; |\tau^3 + 1| > r\}.$$

Again ∞ is in the domain. The points $-1, e^{i\pm\frac{\pi}{3}} = c_1, c_2, c_3$ belong to three simply connected domains D_j with real analytic boundary $\gamma_1, \gamma_2, \gamma_3$. The union of their (pairwise disjoint) closures constitute the complement of Ω_2 in \mathbb{P}^1 . The origin and infinity are in the interior of the domain.

We translate 1 unit to the right:

$$\Omega_3 := \{\sigma \in \mathbb{P}^1; |(\sigma - 1)^3 + 1| > r\}$$

The points $0, 1 + e^{\pm i\frac{\pi}{3}} = c'_1, c'_2, c'_3$ belong to three simply connected domains D'_j with real analytic boundary $\gamma'_1, \gamma'_2, \gamma'_3$. The union of their (pairwise disjoint) closures constitute the complement of Ω_3 . The point 1 and infinity belong to the interior of Ω_3 . Also $c' = 0$ is contained in a neighborhood D'_1 which is in the complement of Ω_3 .

Next we make an inversion so that the domain becomes bounded.

$$\Omega_4 = \Omega = \{w \in \mathbb{P}^1; |\left(\frac{1}{w} - 1\right)^3 + 1| > r\}.$$

The points $\infty, \frac{1}{1+e^{\pm i\frac{\pi}{3}}} = c''_1, c''_2, c''_3$ belong to three simply connected domains D''_j with real analytic boundary $\gamma''_1, \gamma''_2, \gamma''_3$. The union of their closures constitute the complement of Ω . The point 1 and 0 belong to the interior of Ω . Ω lies inside the curve γ''_1 and there are two holes, bounded by γ''_2, γ''_3 respectively.

We rewrite the description of Ω :

$$\Omega = \{w \in \mathbb{P}^1; \log \left| \left(\frac{1}{w} - 1\right)^3 + 1 \right| - \log r > 0\}$$

$$\Omega = \{w \in \mathbb{P}^1; \log \left| \frac{(1-w)^3 + w^3}{w^3} \right| - \log r > 0\}$$

$$\Omega = \{w \in \mathbb{P}^1; \log \left| \frac{w^3}{(1-w)^3 + w^3} \right| + \log r < 0\}$$

$$\Omega = \{w \in \mathbb{P}^1; \log \left| \frac{w^3}{1-3w+3w^2} \right| + \log e^{-1/3} < 0\}$$

$$\Omega = \{w \in \mathbb{P}^1; \frac{1}{3} \log \left| \frac{w^3}{1-3w+3w^2} \right| + \frac{1}{3}(-1/3) < 0\}$$

Let $G(w) = \frac{1}{3} \log \left| \frac{w^3}{1-3w+3w^2} \right| - \frac{1}{9}$. Then Ω is the set where this function is negative. In a neighborhood of the origin, we have $G(w) = \log |w| + \mathcal{O}(1)$. In the rest of Ω the function is harmonic. Hence $G(w)$ is the Green function for Ω with a pole at the origin.

Next we consider the function G in a neighborhood of the interior point $w = 1$: We get $G(1) = -\frac{1}{9}$. We estimate $G(w) + \frac{1}{9}$:

$$\begin{aligned}
G(w) + \frac{1}{9} &= \frac{1}{3} \log \left| \frac{w^3}{w^3 + (1-w)^3} \right| \\
&= \frac{1}{3} \log \left| \frac{1}{1 + \left(\frac{1-w}{w}\right)^3} \right| \\
&= \frac{1}{3} \log \left| 1 - \left(\frac{1-w}{w}\right)^3 + \mathcal{O}\left(\left(\frac{1-w}{w}\right)^6\right) \right| \\
&= \frac{1}{3} (\operatorname{Re})(\log(1 - \left(\frac{1-w}{w}\right)^3 + \mathcal{O}\left(\left(\frac{1-w}{w}\right)^6\right))) \\
&= \frac{1}{3} (\operatorname{Re})\left(-\left(\frac{1-w}{w}\right)^3 + \mathcal{O}\left(\left(\frac{1-w}{w}\right)^6\right)\right) \\
&= 3(\operatorname{Re})(-(1-w)^3 + \mathcal{O}(1-w)^4) \\
&= \mathcal{O}(|w-1|^3)
\end{aligned}$$

□

Lemma 2.2. *Let S_ϵ denote the sector $S_\epsilon = \{w = 1 + re^{i\theta}; 0 < r < \epsilon^{1/3}/2, \frac{11\pi}{12} < \theta < \frac{13\pi}{12}\}$. There is an $\epsilon_0 > 0$ so that if $0 < \epsilon < \epsilon_0$ then*

$$S_\epsilon \subset \{w \in \Omega; -\epsilon < G(w) - G(1) < 0\}.$$

Proof. Suppose that $w \in S_\epsilon$. Then

$$\begin{aligned}
G(w) - G(1) &= G(w) + \frac{1}{9} \\
&= \frac{1}{3}(Re)(w-1)^3 + \mathcal{O}(w-1)^4 \\
&\Rightarrow \\
\frac{1}{3}(Re)((w-1)^3) - Cr^4 &\leq G(w) - G(1) \\
&\leq \frac{1}{3}(Re)((w-1)^3) + Cr^4 \\
&\Rightarrow \\
3r^3 \cos(3\theta) - Cr^4 &\leq G(w) + 1 \\
&\leq 3r^3 \cos(3\theta) + Cr^4 \\
2\pi + \frac{3}{4}\pi &\leq 3\theta \\
&\leq 2\pi + \frac{5}{4}\pi \\
&\Rightarrow \\
-1 &\leq \cos(3\theta) \\
&\leq -\sqrt{2}/2 \\
-\frac{1}{3}r^3 - Cr^4 &\leq G(w) - G(1) \\
&\leq -\frac{1}{3}\sqrt{2}/2r^3 + Cr^4 \\
-\frac{\epsilon}{24} - C\frac{\epsilon^{4/3}}{16} &\leq G(w) - G(1) \\
&\leq -\frac{\sqrt{2}\epsilon}{48} \\
&\Rightarrow \\
-\epsilon &< G(w) - G(1) < 0
\end{aligned}$$

□

Define the function $s(t)$ for $-\infty < t < 0$.

$$s(t) = \lambda(\{G(w) < t\})$$

i.e. the area of the sublevel set where $G < t$.

Corollary 2.3. *For all small enough $\epsilon > 0$ we have that*

$$s(-\frac{1}{9}) - s(-\frac{1}{9} - \epsilon) \geq \pi \frac{\epsilon^{2/3}}{48}.$$

Proof. We have that

$$\begin{aligned} s(-\frac{1}{9}) - s(-\frac{1}{9} - \epsilon) &= \lambda(\{-\frac{1}{9} - \epsilon < G < -\frac{1}{9}\}) \\ &\geq \lambda(S_\epsilon) \\ &= \pi(\frac{\epsilon^{1/3}}{2})^2 \frac{1}{12} \end{aligned}$$

This implies that

$$s(-\frac{1}{9} - \epsilon) \leq s(-\frac{1}{9}) - \pi \frac{\epsilon^{2/3}}{48}.$$

□

The following is immediate.

Theorem 2.1. *The function is not convex.*

Corollary 2.4. *The function $t \rightarrow \log \lambda(\{G < t\})$ is not convex.*

REFERENCES

1. Blocki, Z., Zwonek, W.; Estimates for the Bergman kernel and the multidimensional Suita conjecture. *New York J. of Math.* **21** (2015), 151–161